Suppression of Inherent Instabilities in Initial Value Methods

T. S. WANG AND D. DOBROTT

General Atomic Company, San Diego, California 92138

A simple integration method is presented that mitigates inherent instabilities in certain ordinary differential equations. These instabilities occur for initial value problems or for problems of infinite range where initial value methods are required. Our technique uses a modified initial value method that employs a restart scheme. The restart values for the dependent variables are found by the Newton–Raphson technique. The method is applied to the "tearing mode" equations of resistive magnetohydrodynamics. Accuracy to six places is achieved.

INTRODUCTION

The purpose of this paper is to describe a simple method of mitigating the effect of undesired exponentially dominant solutions occurring in the numerical integration of certain ordinary differential equations. This method is particularly appropriate for integration over infinite ranges and may be used in conjunction with conventional integration method such as Runge–Kutta.

In particular, we consider the integration of the differential equation

$$y'' - x^2 y = f(x) \tag{1}$$

over the range $-\infty < x < \infty$. Here we consider $f(x)$ to be of bounded variation for finite x. This equation occurs in the study of the "tearing instability" in resistive, nonideal magnetohydrodynamics [1].

We look for bounded solutions of Eq. (1), for all x, subject to a specification of $y(0)$. This boundedness condition may not be applied a priori in a numerical integration scheme. Instead, an initial value method may be used that specifies $y(0)$ and $y'(0)$. These initial values are then adjusted so that solutions are bounded for large $|x|$. That this method is ill-posed may be seen by noting that the homogeneous solution to Eq. (1) is

$$y(x) = A\, x^{1/2} K_{1/4}(x^2/2) + B\, x^{1/2} I_{1/4}(x^2/2). \tag{2}$$

Here $K_{1/4}(z)$ and $I_{1/4}(z)$ are modified Bessel functions. The first term in Eq. (2) may be eliminated in numerical integration by the condition, $y(0) = 0$. However, the second term is ever present and is proportional to $\exp(x^2/2)/x^{1/2}$ for large x.

93

A shooting method may be used to suppress this undesired solution, but one must make $|B| < 10^{-16}$ to obtain three-place accuracy for $|x| < 5$. Minimization of $|B|$ has been achieved to this extent by adjusting the initial values, $y(0)$ and $y'(0)$ [2]. Such a procedure is limited by the machine accuracy and the absolute minimum expected may be given approximately by $\min|B| \sim 10^{-1}$, where l is the number of digits of precision of the computer. From Eq. (2) it is clear that the absolute error from the integration due to the undesired exponential term is of the order $|B|$ $\exp(x^2/2)/x^{1/2}$. If a prescribed accuracy of m digits is given, the range of x is correspondingly limited by an upper bound X, where X is determined from

$$|B| \cdot (e^{X^2}/2)/X^{1/2} \simeq y(X) \cdot 10^{-m}.$$

Certain backward integration methods [3] require precise knowledge of the dependent variable for large x. Small errors in the starting values grow rapidly and produce grossly inaccurate solutions for small x.

It has been suggested that such difficulties that arise from inherent instabilities may be overcome by means of the "partial Wronskian" method [4]. However, we have found that this technique inevitably fails because of accumulated errors and errors due to curve-fitting procedures.

We have been able to suppress these effects and achieve greater accuracy over an extended range by a modification of the initial value process. We simply select a value $x_1 > 0$ such that $|B| \exp(x_1^2/2)/|x_1|^{1/2}$ is smaller than the desired accuracy and restart the integration at $x = x_1$. We then minimize $|B|$ as function of $y(x_1)$ and $y'(x_1)$ by the Newton–Raphson method. Solutions with uniform six-place accuracy may be obtained for all x by appending as asymptotic solutions for $|x| > 10$.

As an application, we apply the procedure to a sequence of problems of the form of Eq. (1) where the right-hand side is given by the solution to the previous problem. This application arises in the study of diffusive effects in tearing modes [5]. Details of the procedure and the application are given in the next section.

MODIFIED INITIAL VALUE SCHEME

A. *Method Outline*

We begin our integration of Eq. (1) with the usual initial value technique. We set $y(0) = 0$ and arbitrarily select $y'(0) < 0$. This latter choice is motivated by the asymptotic form of the solution, which may be determined directly from the differential equation. Eventually, the integration is swamped by the undesired exponential solution and $|B|$ may be computed by curve fitting. The value of $|B|$ clearly depends upon $y'(0)$. Subsequent modification of $y'(0)$, $\delta y'(0)$, is made by means of Newton's method to minimize $|B|$, i.e.,

$$\delta y'(0) = -|B|/[\partial|B|/\partial y'(0)]. \tag{3}$$

This technique yields a solution of prescribed accuracy for a truncated range $0 \leqslant |x| \leqslant x_1$ (see Fig. 1, curve A).

To extend the range beyond x_1, we restart the integration of Eq. (1) at x_1 and adjust $y(x_1)$ and $y'(x_1)$ to minimize $|B|$ by the "gradient method" [6]. Since the bounded solution of Eq. (1) is unique and corresponds to $|B| = 0$, we may visualize nested nontrivial $|B|$ curves, as shown in Fig. 2, as functions of $y(x_1)$ and $y'(x_1)$. In order to find the minimum $|B|$ first we expand $|B|$ at point 1 (where $|B| \equiv 0$) as a Taylor series about point 2, so that

$$|B|_n + \delta y'_{n+1}(x_1) \frac{\partial |B|_{n+1}}{\partial y'(x_1)} + \delta y_{n+1}(x_1) \frac{\partial |B|_{n+1}}{\partial y(x_1)} = 0. \tag{4}$$
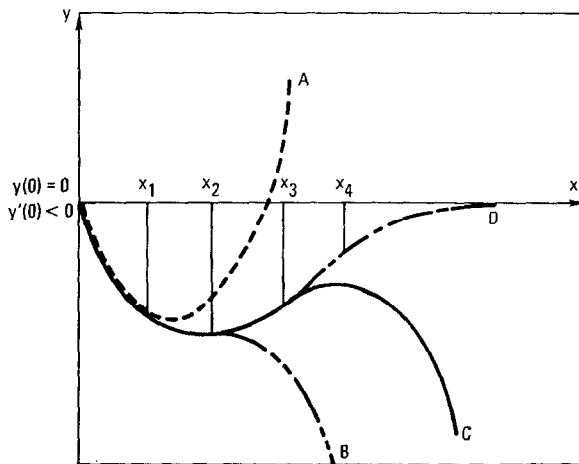

FIG. 1. Application of the modified initial value method. The figure shows extension of the numerical range by three applications of this restart technique (curves A, B, and C) and continuation to an asymptotic solution (curve D).

The proper values of $\delta y(x_1)$ and $\delta y'(x_1)$ are approximated by projecting a normal from point 2, as seen in Fig. 2. Since the equation for the normal is given by

$$\delta y_{n+1}(x_1) \frac{\partial |B|_{n+1}}{\partial y'(x_1)} - \delta y'_{n+1}(x_1) \frac{\partial |B|_{n+1}}{\partial y(x_1)} = 0, \tag{5}$$

δy_{n+1} and $\delta y'_{n+1}$ are obtained from Eqs. (4) and (5). Convergence is assured if the neighborhood of point 1 is sufficiently small.

The whole process, therefore, consists of five steps. First we obtain a solution to the desired accuracy by using Newton's method for the range $0 \leqslant x \leqslant x_1$. Second we successively vary $y(x_1)$ and $y'(x_1)$ slightly and obtain new values of $|B|$ by integrating beyond $x = x_1$. Next we can compute gradients of $|B|$ with respect to $y(x_1)$ and $y'(x_1)$. We use these gradients to compute $\delta y_{n+2}(x_1)$ and $\delta y'_{n+1}(x_1)$ from Eqs. (4) and (5)

and, hence, new restart values in the fourth step. Finally, in step 5, integration of Eq. (1) is continued with the new restart values

$$y_{n+2}(x_1) = y_{n+1}(x_1) + \delta y_{n+1}(x_1)$$

and

$$y'_{n+2}(x_1) = y_{n+1}(x_1) + \delta y_{n+1}(x_1).$$

By looping through steps 2–5, $| B |$ is minimized to a prescribed degree and we obtain the desired accuracy for the range $x_1 \leqslant | x | \leqslant x_2$ (see Fig. 1, curve B).
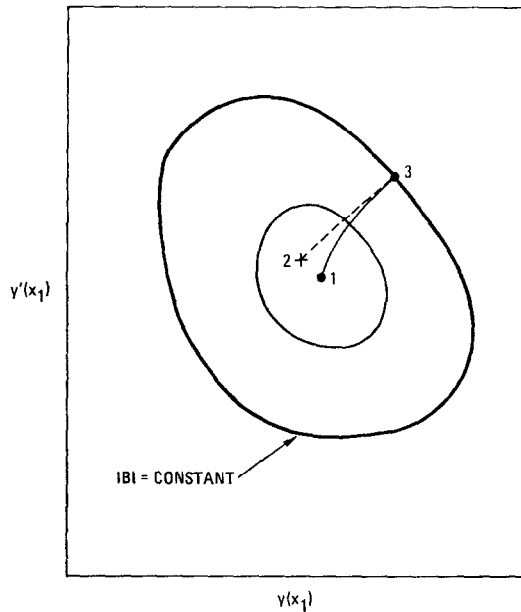

FIG. 2. Gradient method. Nested constant $| B |$ curves are shown as functions of restart values $y(x_1)$ and $y'(x_1)$. Note that $| B | = 0$ at point 1. Approximate minimizations at point 3 are found by normal projections from point 2.

Because the starting point has been moved from the origin to x_1, the expected absolute minimum $| B |$ is given approximately by $(\min | B |) \exp[(x_1^2/2)/| x_1 |^{1/2}] \sim 10^{-1}$. For $x_1 = 5$, $\min | B |$ may be reduced by five orders of magnitude. Usually, three to four orders of magnitude improvement can be achieved easily. Clearly the method may be repeated at larger x values as many times as necessary (Fig. 1, curve C). The restarting procedure may be automated or may be interactive. When an accurate solution is obtained at sufficiently large x, a uniform solution may be obtained for all $| x | > 10$ by appending an asymptotic solution (Fig. 1, curve D).

B. *Application of Restart Scheme*

A demonstration of the absolute accuracy attainable may be seen by integration of Eq. (1) when $f(x) = 0$. Then the desired solution is

$$y(x) = x^{1/2} K_{1/4}(x^2/2)$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \frac{x}{\Gamma(3/4)} \int_0^\infty dt \, \sinh^{1/2} t \, \exp\left(-\frac{x^2}{2} \cosh t\right). \tag{6}$$

The modified initial value scheme was run interactively in this case and the results are compared in Table I with those obtained from Eq. (6). Absolute accuracy of 10^{-15} is noted. Such accuracy using direct inversion matrix methods is not possible.

TABLE I

Absolute Accuracy. Results from Eq. (6), the Initial Value, and the Modified Initial Value Methods are Compared

x	Eq. (6)	Uncorrected[a]	Corrected[b]
1	$9.603E - 01$	$9.603E - 01$	$9.603E - 01$
2	$1.632E - 01$	$1.632E - 01$	$1.632E - 01$
3	$1.116E - 02$	$1.114E - 01$	$1.116E - 02$
4	$2.940E - 04$	$-1.572E - 05$	$2.940E - 04$
5	$2.933E - 06$	$-2.482E - 02$	$2.933E - 06$
6	$1.097E - 08$	$-5.531E + 00$	$1.097E - 08$
7	$1.528E - 11$	$-3.401E + 03$	$1.529E - 11$
8	$7.913E - 15$	$-5.746E + 06$	$7.977E - 15$

[a] $| B | = 2.056E - 07.$
[b] $| B | = 1.675E - 30.$

Six-place accuracy is obtainable for the tearing mode application, where we take

$$f(x) = x. \tag{7}$$

We look for bounded solutions with $y(0) = 0$ and $y'(0) \neq 0$. For this choice of $f(x)$, the solution may be expressed as a quadrature [7],

$$y(x) = -\frac{x}{2} \int_0^{\pi/2} d\theta \, \sin^{1/2} \theta \, \exp\left(-\frac{x^2}{2} \cos \theta\right). \tag{8}$$

The solution, Eq. (8), may be verified directly by substitution and integration by parts. We may use Eq. (8) to check the accuracy of our modified initial value technique.

A uniformly accurate solution is obtained for all x by appending an asymptotic solution for $|x| > 10$. We find directly from Eqs. (1) and (7) that

$$y \sim -\frac{1}{x} - \frac{2}{x^3} - \frac{60}{x^9} + \cdots .$$ (9)

The results are shown in Table II, where the initial value method is compared to Eq. (8). Six-place accuracy is noted.

TABLE II

The Values of y Computed from Eq. (8) and the Modified Initial Value Method; (MIVM)

x	Eq. (8)	MIVM
0	$0.000000E + 00$	$0.000000D + 00$
1	$-0.458771E + 00$	$-0.458771D + 00$
2	$-0.471784E + 00$	$-0.471784D + 00$
3	$-0.339902E + 00$	$-0.339903D + 00$
4	$-0.252187E + 00$	$-0.252187D + 00$
5	$-0.200676E + 00$	$-0.200676D + 00$
6	$-0.166930E + 00$	$-0.166930D + 00$
7	$-0.142978E + 00$	$-0.142978D + 00$
8	$-0.125062E + 00$	$-0.125062D + 00$
9	$-0.111145E + 00$	$-0.111145D + 00$
10	$-0.100020E + 00$	$-0.100020D + 00$
11	$-0.909216E - 01$	$-0.909215D - 01$
12	$-0.833414E - 01$	$-0.833414D - 01$
13	$-0.769285E - 01$	$-0.769285D - 01$
14	$-0.714323E - 01$	$-0.714323D - 01$
15	$-0.666694E - 01$	$-0.666694D - 01$

Another application, where this accuracy is required, arises from the study of the effects of diffusion on the tearing mode [5]. In this circumstance, it is found that the displacement resistive layer for the case of a resistive sheet pinch satisfies the equation

$$cy''' + y'' - x^2y = x - c.$$ (10)

Here c is proportional to the diffusion velocity. Since the parameter c is a coefficient of the highest derivative, Eq. (10) represents a singular perturbation problem, even

for small c. Nevertheless, Eq. (1) may be solved for bounded solutions by expanding y in powers of c and solving the resulting hierarchy

$$h_0'' - x^2 h_0 = x,$$

$$h_1'' - x^2 h_1 = -h_0''' - 1, \qquad (11)$$

$$h_2'' - x^2 h_2 = -h_1'''$$

in succession. Here we have written

$$y = h_0 + c h_1 + c^2 h_2 + \dots . \qquad (12)$$
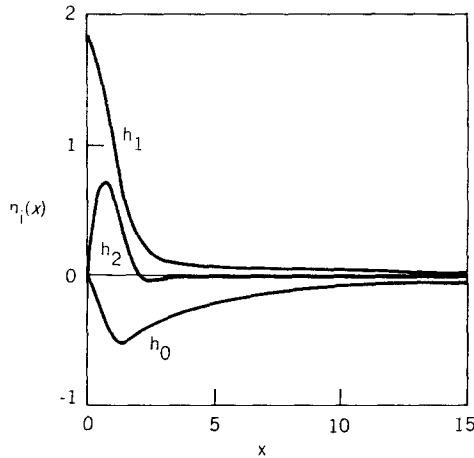

FIG. 3. h_i versus x for $i = 1, 2, 3$.

Asymptotic expressions for h_1 and h_2 are inferred from that for h_0 so that

$$h_1 \sim \frac{1}{x^2} + \frac{12}{x^6} + \frac{924}{x^{10}} + \frac{161{,}040}{x^{14}} + \dots .$$

and

$$h_2 \sim -\frac{24}{x^7} - \frac{5{,}376}{x^{11}} - \frac{1{,}929{,}312}{x^{15}} + \dots . \qquad (13)$$

Cubic splines are used to obtain values of the right-hand side of Eq. (11) necessary in the differential equation algorithm. The results for h_0, h_1, and h_2 are shown in Fig. 3 for $0 \leqslant x \leqslant 15$.

Conclusion

The correction procedure described here is more or less independent of the accumulated errors in the previous step. In principle, one can find the exact solution by starting the integration at any x. Therefore, the advantage of having an approximate solution at, say, x_1, is to narrow the parameter search area for min $\mid B \mid$. The better the solution, the smaller the parameter space needed to be searched. This particular property makes the procedure much more attractive and gives much more accurate answers than other methods. For example, the partial Wronskian method often fails completely due to accumulated errors and errors arising due to curve-fitting procedure. A consequence of this self-correcting property is that the integration is not required to have extreme accuracy at the beginning of the integration in order to obtain certain predetermined accuracy at the end point. Therefore, more uniformed accuracy can be obtained over a larger integration range.

References

1. H. P. Furth, J. Killeen, and M. N. Rosenbluth, *Phys. Fluids* 6 (1963), 459.
2. B. Coppi, J. M. Greene, and J. L. Johnson, *Nucl. Fusion* 6 (1966), 101.
3. L. Fox, "Numerical Solution of Ordinary and Partial Differential Equations," p. 84, Addison-Wesley, Reading, Mass., 1962.
4. J. P. Boris and J. M. Greene, *J. Computational Physics* 4 (1969), 30–42.
5. D. Dobrott, S. C. Prager, and J. B. Taylor, *Phys. Fluids* 20 (1977).
6. R. W. Haming, "Introduction to Applied Numerical Analysis," p. 226, McGraw–Hill, New York 1971.
7. R. D. Hazeltine, D. Dobrott, and T. S. Wang, *Phys. Fluids* 18 (1975), 1783.